9. Zatsepin, Iu. A., Popov, E. G. and Tsikulin, M. A., Luminescence of shock wave fronts in certain gases. ZhETF Vol. 54, №1, 1968.
10. Grim, G., Spectroscopy of Plasma. M., Atomizdat., 1969.
11. Onufriev, A. T. and Sevast'ianenko, V. G., Radiative transfer in spectral lines with self-absorption. PMTF, №2, 1966.
12. Afanas'ev, V. V., Krol', V. M., Krokhin, O. N. and Nemchinov, I. V., Gasdynamic processes in heating of a substance by laser radiation. PMM Vol. 30, №6, 1966.
13. Krol', V. M. and Nemchinov, I. V., Self-similar motion of a gas heated by nonequilibrium continuous radiation spectrum. PMTF, №5, 1968.
14. Nemchinov, I. V., Expansion of a plane gas layer with gradual energy release. PMTF, №1, 1961.
15. Germogenova, T. A. and Sushkevich, T. A., Solution of the transport equation by the method of average fluxes. Problems in the Physics of Reactor Protection. M., Atomizdat, №3, 1969.
16. Gol'din, V. A., Danilova, G. V. and Chetverushkin, B. N., Approximate calculation method for the unsteady kinetic equation. Collection "Computational Methods in the Transport Theory". M., Atomizdat, 1969.

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DISCONTINUITY SURFACES IN DISPERSE SYSTEMS

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The equations of conservation of mass and momentum are considered for a fluid dispersing medium with suspended particles (the dispersible phase) along an arbitrary discontinuity surface of a disperse system. Conditions binding the velocity, pressure and concentration jumps are derived, and a model of surface tension at such surface is suggested. The coefficient of surface tension is dependent on the interrelation between the densities of phases, size of the dispersed phase particles, as well as on other parameters.

The problem of stability of the horizontal surface of a concentration discontinuity is solved. It is shown that, when a suspended layer is above the discontinuity surface, this surface is stable with respect to perturbations of sufficiently small wave length. The critical wave length, which defines the limit conditions of the onset of piston type fluidization, substantially depends on the effective surface tension. The upper free surface of the suspended layer remains, as expected, stable relative to perturbations of any wave length. The obtained results are in agreement with available experimental data.

A number of problems of mechanics of disperse systems reduce to the investigation of discontinuity surfaces. One of the most important among these is the determination of conditions for the occurrence of piston type fluidization which disrupts in the system the regular pattern of technological processes [1, 2]. The piston mode implies a sharp disruption of the suspended layer homogeneity, and can only be observed in sufficiently narrow tubes. It is characterized by a vertical stratification of a two-phase system into layers containing either the two-phase mixture or the homogeneous dispersing medium, thus resulting in a multiple-sandwich-like structure of the whole system. It is natural to expect that the problem of existence of the piston mode can be reduced to the analisis of stability of the discontinuity surface.

Conditions at the discontinuity surface in a disperse system were, apparently, first considered in [3, 4]. An attempt at solving the problem of the discontinuity surface stability made by Rice and Wilhelm [5] led them to the conclusion that there was complete instability at the discontinuity surface in the case of the mixture lying above a homogeneous dispersing medium. The discrepancy between experimental data and the conclusion reached by these authors results from the use of a crude model for the suspended layer and for the discontinuity surface. The same model was later used by Murray [6] for proving the stability of the upper free surface of a suspended layer. The main shortcoming of these investigations was the neglect of surface tension.

The existence of surface tension on a surface of discontinuity in a disperse system was itself a subject of controversy [1, 2]. As far as the writers are aware, neither the supporters, nor the opponents of this hypothesis had given any substantiated reasons for their respective views, if one disregards references to the similarity of a number of phenomena in a suspended layer and in a homogeneous fluid [2, 7]. There had even been an attempt at experimental determination of the surface tension coefficient [7].

In this paper the discontinuity surface stability in a suspended layer is considered on the basis of strict initial conditions obtaining at this surface including, in particular, a certain model of surface tension.

1. Conditions at the surface of discontinuity. Model of surface tension. As the initial model of the disperse system we take a binary medium consisting of two interpenetrating and interacting continuous media. For simplicity, we assume the two media to be perfect fluids, i.e. we shall neglect stress deviators in both of these. The equations of conservation of momentum and mass of the dispersing medium and of the dispersed phase may then be written as

$$d_1\varepsilon\left[\frac{\partial}{\partial t} + (\mathbf{v}\nabla)\right]\mathbf{v} = -\nabla p_1 + d_1\varepsilon\mathbf{g} - \mathbf{f}$$

$$d_2\rho\left[\frac{\partial}{\partial t} + (\mathbf{w}\nabla)\right]\mathbf{w} = -\nabla p_2 + d_2\rho\mathbf{g} + \mathbf{f} \qquad (1.1)$$

$$\frac{\partial\varepsilon}{\partial t} + \nabla(\varepsilon\mathbf{v}) = 0, \qquad \frac{\partial\rho}{\partial t} + \nabla(\rho\mathbf{w}) = 0, \qquad \varepsilon + \rho = 1$$

Here $\mathbf{v}$, d_1, p_1 and $\mathbf{w}$, d_2, p_2 are, respectively, the velocities, densities, and pressures of the dispersing medium and of the dispersed phase, ρ is the volume concentration of the dispersed phase ($\varepsilon = 1 - \rho$ is the system porosity), $\mathbf{f}$ is the force of interaction between the dispersing medium and the dispersed phase, and $\mathbf{g}$ is the acceleration due to the external field forces.

We represent the interaction force $\mathbf{f}$ in the form

$$\mathbf{f} = -\rho\nabla p_1 - \rho\mathbf{u}F(u, \rho), \qquad \mathbf{u} = \mathbf{w} - \mathbf{v} \qquad (1.2)$$

where the first term corresponds to the force acting on the particles of the dispersed phase caused by pressure gradient in the dispersing medium, and the second term – to the generally nonlinear resistence force acting on the particles during their motion relative to the dispersing medium. Both forces relate to a unit volume of the mixture.

Expressing force **f** in the form (1.2) implies the taking into account its main components only, while neglecting the effects of acceleration in the relative motion of particles (the effect of apparent additional mass and increased resistance due to unsteady flow past the particles). This is fully justified, e.g. in the case of particles of considerably greater density than that of the dispersing medium, or of motion whose frequency is small in comparison with the reciprocal of the particle velocity relaxation time in the stream.

Let us now assume that within the volume of a disperse system there is a certain discontinuity surface on both sides of which the concentration of particles is different. It is natural to assume that this concentration jump will be accompanied by discontinuities of other parameters.

Such surface may separate regions containing particles of one kind and, in particular, a disperse system from the stream of a homogeneous dispersing medium. It can also serve as the interface of regions containing particles of different kinds, which is characteristic, for example, of problems related to the separation of particles in a stream.

The discontinuity surface is clearly an idealized concept of an actual transition layer of thickness 2Δ in which parameters of the disperse system vary much more abruptly than in the regions on both of its sides. Clearly, the order of magnitude of Δ is the same as that of the mean distance between particles in the neighborhood of that layer. However, the disperse system model defined by the continuity equations (1.1) is, strictly speaking, valid for investigating processes whose characteristic linear scale is considerably greater than the indicated distance. Hence, in the framework of this model we can set Δ tending to zero, and talk of a surface of discontinuity.

We introduce a system of orthogonal coordinates x, y, z attached to a certain element of plane $z = 0$ of the discontinuity surface S, and denote the values in regions $z > 0$ and $z < 0$ by superscripts plus and minus, respectively.

Integrating the mass conservation equation (1.1) with respect to z from $-\Delta$ to $+\Delta$ and decreasing Δ to zero, we obtain for the values on the two sides of the surface $z = 0$ the following relationships:

$$(\varepsilon v_z)^+ = (\varepsilon v_z)^-, \qquad (\rho w_z)^+ = (\rho w_z)^- \tag{1.3}$$

which represent the conditions of continuity of the streams of the fluid and particles across surface S.

To obtain conditions for the tangential velocity components, we integrate in a similar manner the x- and y-components of the momentum conservation equations (1.1). For this we first transform the left-hand sides of these with the aid of the equation of mass conservation as follows:

$$\varepsilon\left[\frac{\partial}{\partial t} + (\mathbf{v}\nabla)\right]\mathbf{v} = \frac{\partial}{\partial t}(\varepsilon\mathbf{v}) + \nabla(\varepsilon\mathbf{T}_v), \ \rho\left[\frac{\partial}{\partial t} + (\mathbf{w}\nabla)\right]\mathbf{w} = \frac{\partial}{\partial t}(\rho\mathbf{w}) + \nabla(\rho\mathbf{T}_w)$$
$$\mathbf{T}_v = \|v_i v_j\|, \qquad \mathbf{T}_w = \|w_i w_j\|, \qquad i,j = x, y, z \tag{1.4}$$

Using transformation (1.4), from the x- and y-components of the momentum conservation equation (1.1) we obtain

$$(\varepsilon v_x v_z)^+ = (\varepsilon v_x v_z)^-, \qquad (\varepsilon v_y v_z)^+ = (\varepsilon v_y v_z)^-$$
$$(\rho w_x w_z)^+ = (\rho w_x w_z)^-, \qquad (\rho w_y w_z)^+ = (\rho w_y w_z)^- \tag{1.5}$$

Relationships (1. 3) and (1. 5) yield

$$v_x^+ = v_x^-, \quad v_y^+ = v_y^-, \quad w_x^+ = w_x^-, \quad w_y^+ = w_y^- \tag{1.6}$$

which represent the conditions of continuity of the tangential velocity components at surface S.

From the z-component of the equation of momentum conservation of the dispersing medium we similarly obtain

$$\tfrac{1}{2} d_1 [(v_z^2)^- - (v_z^2)^+] = p_1^+ - p_1^- \tag{1.7}$$

It is more convenient to consider the sum of the z-components of the momentum conservation equation of the dispersed phase and of that of the dispersing medium, rather than the z-component of the momentum conservation equation of the dispersed phase. Then, using again transformation (1. 4), we obtain

$$d_1 [(\varepsilon v_z^2)^- - (\varepsilon v_z^2)^+] + d_2 [(\rho w_z^2)^- - (\rho w_z^2)^+] = p_1^+ - p_1^- + p_2^+ - p_2^- \tag{1.8}$$

The relationships (1. 7) and (1. 8) obviously represent, respectively, the conditions of equilibrium of normal stresses in the dispersing medium and of total normal stresses of the whole system along surface S. The equation of equilibrium of the total tangential stresses directly follows from relationships (1. 5). These, together with condition (1. 8), may be considered as the conditions of the density tensor continuity for the total stream of momentum $\mathbf{\Pi}$ at surface S

$$\mathbf{\Pi} = (p_1 + p_2)\mathbf{I} + d_1 \varepsilon \mathbf{T}_v + d_2 \rho \mathbf{T}_w, \qquad \mathbf{I} = \|\delta_{ij}\| \tag{1.9}$$

We note that condition (1. 7) may, also, be derived directly from the Bernoulli integral for the flow of liquid through a lattice of particles. However, in the derivation of (1. 7) the conditions necessary for the existence of the Bernoulli integral were not used.

We would emphasize that relative velocities appear in the conditions derived above for the discontinuity surface, and that, generally, the normal velocity components are not necessarily zero, in other words, a migration of particles through the discontinuity surface is possible. However, in a number of cases, for example, when S is the boundary between a disperse system and a homogeneous dispersing medium, such migration of particles is not possible, since from the second of conditions (1. 3) follows that $w_z^- = 0$ when $\rho^+ = 0$.

Only a plane element of the discontinuity surface was considered above. Let us now see how condition (1. 8) is to be altered in the case of total normal stresses at a curvilinear interface $z = \zeta(t, x, y)$. For this we will consider the work δA required for the virtual dislocation $\delta\zeta$ of this interface. The work δA is, obviously, the sum of the work expended on changing the volume of the disperse system and of that of changing the area δS of the discontinuity surface S.

Then

$$\delta A = \int_S \{p_1^+ + p_2^+ - p_2^- - p_2^- + d_1 [(\varepsilon v_z^2)^- - (\varepsilon v_z^2)^+] + d_2 [(\rho w_z^2)^- - (\rho w_z^2)^+]\} \delta\zeta \, dS + \alpha \delta S \tag{1.10}$$

where α is the energy required for a unit increase of the discontinuity surface area.

The reasoning which follows is exactly the same as that applicable to the interface of two monophase fluids (see, e. g. [8]). As the result we obtain for the total normal stresses

$$p_1^+ - p_1^- + p_2^+ - p_2^- + d_1[(\varepsilon v_z^2)^- - (\varepsilon v_z^2)^+] + d_2[(\rho w_z^2)^- - (\rho w_z^2)^+] - \\ - \alpha(R_1^{-1} + R_2^{-1}) = 0 \qquad (1.11)$$

where R_1 and R_2 are the principal radii of curvature of the discontinuity surface.

It will be seen from (1.11) that in the case of a curvilinear surface the condition for the total normal stresses differs from the corresponding condition (1.8), applicable to a plane surface, by a supplementary term which is due to the surface curvature and normal translation of particles.

Obviously, the conditions for total tangential stresses and for normal tensions in the fluid, as well as those for the continuity of streams of fluid and particles remain unchanged, hence, conditions (1.3), (1.6) and (1.7) are valid for a curvilinear discontinuity surface.

Relationships (1.3), (1.6), (1.7) and (1.11) represent the boundary conditions imposed on the solution of the system of Eqs. (1.1) at the discontinuity surface $z = \zeta(t, x, y)$.

Let us now consider parameter α, appearing in condition (1.11) and representing the coefficient of effective surface tension at the discontinuity surface, as the energy required for a unit increase of the discontinuity surface area, as defined in formula (1.10). For this we revert to the 2Δ thick transition layer mentioned above, and in which the parameters of motion in the system are subject to abrupt variations. An increase of the discontinuity surface, obviously, indicates the migration of a definite number of particles from the depth of the disperse system into this transition layer. The work expended on pushing a single particle from the plus region to the center of such layer is

$$A^+ = \lim_{\Delta\to 0} \int_{\Delta}^{+0} \theta^+ \frac{\partial p_1^+}{\partial z} dz = -\frac{\theta^+}{2}(p_1^+ - p_1^-) \qquad (1.12)$$

Moving a particle from the minus region to the center of the transition layer necessitates, similarly, the work

$$A^- = \lim_{\Delta\to 0} \int_{-\Delta}^{-0} \theta^- \frac{\partial p_1^-}{\partial z} dz = \frac{\theta^-}{2}(p_1^+ - p_1^-) \qquad (1.13)$$

where θ^+ and θ^- are the volumes of particles in the plus and minus regions, respectively.

By definition

$$\alpha = A^+N^+ + A^-N^- \qquad (1.14)$$

where N^+ and N^- are the numbers of particles which, owing to the unit increase of the discontinuity surface area, reach it from the plus and minus regions, respectively.

Values N^+ and N^- are readily expressed in terms of volume and concentration of particles

$$N^+ = \left(\frac{\rho^+}{\theta^+}\right)^{2/3}, \quad N^- = \left(\frac{\rho^-}{\theta^-}\right)^{2/3} \qquad (1.15)$$

Now, from (1.12) – (1.15) we finally obtain

$$\alpha = {}^1/_2\,[(\rho^{2/3}\theta^{1/3})^- - (\rho^{2/3}\theta^{1/3})^+]\,(p_1^+ - p_1^-) \qquad (1.16)$$

Thus the coefficient of surface tension is shown to be proportional to the pressure jump at the discontinuity surface of a disperse system.

With the use of condition (1.7) α can, also, be presented in the form

$$\alpha = {}^1/_4 d_1\,[(\rho^{2/3}\theta^{1/3})^- - (\rho^{2/3}\theta^{1/3})^+]\,[(v_z^2)^- - (v_z^2)^+] \qquad (1.17)$$

It can be seen from (1.17) that the coefficient of surface tension reaches its maximum when the discontinuity surface is the interface of a disperse system (e.g. a suspended layer) and a homogeneous dispersing medium. To avoid ambiguity, let us assume that the discontinuity surface is horizontal, and relate the minus superscript to the disperse system, which is assumed homogeneous. Then, taking into consideration the first of conditions (1.3), we obtain $\rho^+ = 0, \quad \rho^- = \rho, \quad \varepsilon^- = \varepsilon, \quad v_z^{\ +} = U, \quad v_z^{\ -} = \varepsilon^{-1} U$

where U is the velocity of the ascending stream of the dispersing medium. In this case the coefficient of surface tension will be

$$\alpha = {}^1/_4 d_1 \rho^{2/3} \theta^{1/3} (\varepsilon^{-2} - 1)\, U^2 \qquad (1.18)$$

To find the dependence of the surface tension coefficient on the physical parameters of the dispersing medium and of particles of the dispersed phase we use the empirical formula proposed in [10] which it is convenient to present in the form

$$U \approx \frac{2}{9}\frac{\nu}{a}\frac{\beta}{1+0.0955\sqrt{\beta}}, \quad \beta = \frac{\sigma a^3 g}{\nu^2}\varepsilon^{19/4}, \quad \sigma = \frac{d_2 - d_1}{d_1} \qquad (1.19)$$

where ν is the kinematic viscosity of the dispersing medium, a is the radius of the dispersed phase particles (particles are assumed to be spherical). This dependence corresponds, also, to the known emperical formulas of Richardson and Zaki [10].

From (1.18) and (1.19) follows that for $\beta \ll 1$ (practically for $\beta < 1$) the coefficient of surface tension is

$$\alpha \approx 0.020 d_1 (1+\varepsilon)(1-\varepsilon)^{5/3} \varepsilon^{15/2} \mu^{-2} (d_2 - d_1)^2 a^5 g^2 \qquad (1.20)$$

i.e. it is inversely proportional to the square of the fluid dynamic viscosity, and directed proportional to the square of the difference between the particle and fluid densities and to the fifth power of the particle size. For $\beta \gg 1$ (practically for $\beta \gtrsim 10^4$)

$$\alpha \approx 2.2 (1+\varepsilon)(1-\varepsilon)^{5/3} \varepsilon^{11/4} (d_2 - d_1)\, a^2 g \qquad (1.21)$$

i.e. in this case the surface tension coefficient is proportional to the difference of densities of particles and fluid and to the square of particle size, while being independent of the fluid viscosity.

Thus, depending on the physical parameters of the disperse system, the coefficient of surface tension can vary between very wide limits.

Let us consider a few typical examples. We shall determine the surface tension coefficient at the horizontal free surface of a homogeneous layer of porosity $\varepsilon = 0.5$ of particles of a catalyst of density $d_2 = 3$ g/cm³ and radius 0.05 and 1 mm fluidized by water ($d_1 = 1$ g/cm³, $\nu = 0.01$ cm²/s) and air ($d_1 = 0.0012$ g/cm³, $\nu = 0.15$ cm²/s). With use of formulas (1.20) and (1.21) we obtain:

for fluidization by water

$U = 0.041$ cm/s, $\alpha = 6 \times 10^{-6}$ erg/cm² ($a = 0.05$ mm)

$U = 4.6$ cm/s, $\alpha = 1.5$ erg/cm² ($a = 1$ mm)

for fluidization by air

$U = 3.2$ cm/s, $\alpha = 4.5 \times 10^{-5}$ erg/cm² ($a = 0.05$ mm)

$U = 190$ cm/s, $\alpha = 3.5$ erg/cm² ($a = 1$ mm)

These values of α prove the groundlessness of the controversy over the existence or otherwise of surface tension in a suspended layer. In fact, this coefficient is negligibly small in the case of minute particles, and appreciable in that of large ones (for comparison we recall that at the water-air interface $\alpha = 72$ erg/cm²).

2. The stability of discontinuity in a suspended layer. Let us investigate the stability of the surface of discontinuity of the solid phase concentration in a suspended layer for the case, most important in practice, in which a horizontal discontinuity surface separates the region of a homogeneous suspended layer from that containing a homogeneous dispersing medium, with the homogeneous fluid stream normal to the discontinuity surface.

This comprises the following two kinds of the discontinuity surface:

1) the stream of fluid is directed toward the suspended layer lying above the dispersing medium (e. g. the upper surface of fluid in the piston mode);

2) the stream of fluid is directed toward the homogeneous dispersing medium lying above a suspended layer (e. g. the upper free surface of a homogeneous suspended layer, or the lower surface of the fluid interlayer in the piston mode of fluidization).

We shall assume for simplicity that the homogeneous dispersing medium and the suspended layer occupy half-spaces, and limit our analysis to the case of linear dependence on $\mathbf{u}$ of the resistance to the relative motion of particles. Function $F(u, \varepsilon)$ in (1.2) assumes then the form [9, 10]

$$F(u, \varepsilon) = K(\rho)\frac{d_2 - d_1}{\tau}, \qquad \tau_0 = \frac{2}{9}\frac{a^2}{\nu}\frac{d_2 - d_1}{d_1} \tag{2.1}$$

where τ is the relaxation time of particle velocity (for spherical particles $\tau = \tau_0$).

Let plane $z = 0$ in an orthogonal system of coordinates x, y, z be the unperturbed discontinuity surface, and let the homogeneous dispersing medium and the suspended layer occupy, respectively, regions $z > 0$ and $z < 0$. As previously, we denote all values in regions $z > 0$ and $z < 0$ by superscripts plus and minus, respectively.

Equations (1.1) with (1.2) and (2.1) taken into account admit the following simple solution:

$$v_x^{\circ+} = v_y^{\circ+} = 0, \quad v_z^{\circ+} = U = -\tau g_z \frac{(\varepsilon^{\circ-})^2}{K(\rho^{\circ-})}; \quad \rho^{\circ+} = 1 - \varepsilon^{\circ+} = 0 \tag{2.2}$$

$$v_x^{\circ-} = v_y^{\circ-} = 0, \ v_z^{\circ-} = U/\varepsilon^{\circ-}, \ w_x^{\circ-} = w_y^{\circ-} = w_z^{\circ-} = 0, \ \rho^{\circ-} = 1 - \varepsilon^{\circ-} = \text{const}$$

$$p_1^{\circ+} = d_1 g_z z + p_{10}^+, \qquad p_1^{\circ-} = [d_1 + (d_2 - d_1)\rho^{\circ-}] g_z z + p_{10}^-$$

$$p_2^{\circ-} = p_{20}^- = (p_{10}^+ - p_{10}^-)\frac{1 - \varepsilon^{\circ-}}{1 + \varepsilon^{\circ+}}, \qquad p_{10}^+ - p_{10}^- = \frac{1}{2} d_1 U^2 \left[\frac{1}{(\varepsilon^{\circ-})^2} - 1\right]$$

which at the discontinuity surface satisfies conditions (1.3) and (1.6) – (1.8), and corresponds to a homogeneous stream of fluid (region $z > 0$) and a stationary homogeneous region of suspended particles (region $z < 0$).

In the expressions (2.2) U is the velocity of the stabilized homogeneous flow of fluid along the z-axis (rate of filtration through a suspended layer of porosity $\varepsilon^{\circ-}$), and p_{10}^+ and p_{10}^-, p_{20}^- are the pressures at the discontinuity surface for $z \to +0$ and $z \to -0$, respectively. Any two of the three parameters U, $\varepsilon^{\circ-}$, τ and, obviously, the densities of the fluid and of the solid phase are assumed to be known.

We note that for the chosen orientation of the z-axis, $g_z = +g$ and $U < 0$ correspond to a discontinuity surface of the first kind, and $g_z = -g$ and $U > 0$ to that of the second kind.

Let us now consider small perturbations of the discontinuity surface

$$z = \zeta(x, y, t)$$

together with related perturbations of the stationary solution (2.2), i. e. we set

$$q(x, y, z, t) = q^\circ(z) + q'(x, y, z, t) \tag{2.3}$$

where q is meant to denote the velocity and pressure components at the two sides of the discontinuity surface. For simplicity it is also assumed that the solid phase concentration is not subject to perturbations, i.e. that perturbations of the discontinuity surface do not affect the homogeneity of the suspended layer; this is in complete agreement with experimental data.

Taking into account the unperturbed solution (2.2), we substitute expressions (2.3) into the Eqs. (1.1) and linearize these with respect to perturbations. In the following, all primes denoting perturbations will be omitted, and, since the solid phase remains unperturbed, all superscripts at $\rho^{\circ-}$ and $\varepsilon^{\circ-}$ will be discarded, i.e. parameters ρ and ε will be taken as the unperturbed values of concentration of the solid phase and of the suspended layer porosity, respectively.

Thus for small perturbations we obtain equations

$$\left(\frac{\partial}{\partial t} + U\frac{\partial}{\partial z}\right)v^+ = -\frac{1}{d_1}\nabla p_1^+, \quad \nabla v^+ = 0, \quad \sigma = \frac{d_2 - d_1}{d_1} \tag{2.4}$$

$$\left(\frac{\partial}{\partial t} + \frac{U}{\varepsilon}\frac{\partial}{\partial z}\right)v^- = -\frac{1}{d_1}\nabla p_1^- + \frac{\sigma\rho K(\rho)}{1-\rho}\frac{w^- - v^-}{\tau}, \quad \nabla v^- = 0$$

$$\rho\frac{\partial w^-}{\partial t} = -\frac{1}{d_2}\nabla(\rho p_1^- + p_2^-) - \frac{\sigma\rho K(\rho)}{1+\sigma}\frac{w^- - v^-}{\tau}, \quad \nabla w^- = 0 \tag{2.5}$$

In the case of perturbations the boundary conditions (1.3), (1.6), (1.7) and (1.11) must be set at the perturbed discontinuity surface $z = \zeta(x, y, t)$. The corresponding conditions related to the unperturbed discontinuity surface $z = 0$ are of the form

$$\frac{\partial\zeta}{\partial t} = w_z^-, \qquad v_z^+ = \varepsilon v_z^- + \rho\frac{\partial\zeta}{\partial t} \tag{2.6}$$

$$v_x^+ = v_x^- + \rho\frac{U}{\varepsilon}\frac{\partial\zeta}{\partial x}, \qquad v_y^+ = v_y^- + \rho\frac{U}{\varepsilon}\frac{\partial\zeta}{\partial y}$$

$$U\left(\frac{1}{\varepsilon^2} - 1\right)\left(v_z^+ - \frac{\partial\zeta}{\partial t}\right) = \frac{1}{d_1}(p_1^+ - p_1^-) - \rho\sigma g_z\zeta$$

$$U\left(\frac{1}{\varepsilon} - 1\right)^2\left(v_z^+ - \frac{\partial\zeta}{\partial t}\right) = \frac{1}{d_1}p_2^- + \frac{\alpha}{d_1}\left(\frac{\partial^2\zeta}{\partial x^2} + \frac{\partial^2\zeta}{\partial y^2}\right)$$

We follow the conventional method of expanding perturbations into spectrum, and analyze the elementary perturbations

$$q' = Q\exp(-i\omega t + \lambda z + ik_x x + ik_y y), \qquad \mathrm{Im}\,k_x = \mathrm{Im}\,k_y = 0 \tag{2.7}$$

Here q' is meant to denote velocity and pressure perturbations on the two sides of the discontinuity surface, as well as the perturbations of this surface itself (in the latter case $\lambda = 0$). A sufficient condition for the instability of the considered discontinuity surface is the presence in the spectrum (2.7) of terms containing $\mathrm{Im}\,\omega > 0$; this condition is also necessary, since system (2.7) is complete, i.e. when $\mathrm{Im}\,\omega < 0$, the discontinuity surface is stable.

For the amplitudes of velocity component perturbations along the x- and y-axes we introduce transformation

$$k_x Q_x + k_y Q_y = kQ, \qquad k = (k_x^2 + k_y^2)^{1/2} \tag{2.8}$$

With the use of (2.8) the problem is reduced to the analysis of two-dimensional perturbations (analogy with the theorem of Squire).

Taking into account transformation (2.8), we substitute (2.7) into Eqs. (2.4) and obtain for the perturbation amplitudes in the region of the dispersing medium the following system of equations:

$$(U\lambda - i\omega) V_z^+ = -\frac{\lambda}{d_1} P_1^+, \quad (U\lambda - i\omega) V^+ = -\frac{ik}{d_1} P_1^+, \quad \lambda V_z^+ + ikV^+ = 0 \tag{2.9}$$

The roots of the characteristic equation of system (2.9) are

$$\lambda_1^+ = -k, \qquad \lambda_2^+ = +k, \qquad \lambda_3^+ = i\omega U^{-1} \tag{2.10}$$

Solutions corresponding to the root λ_2^+ must be discarded, since they do not attenuate at infinity (for $z \to \infty$). It is not difficult to see that the solutions corresponding to root λ_3^+ are attenuated at infinity only for $\mathrm{Im}\,\omega < 0$, $U < 0$, or for $\mathrm{Im}\,\omega > 0$, $U > 0$ Hence in the case of a discontinuity surface of the first kind ($U < 0$) these solutions can also be discarded as being obviously stable. Solutions corresponding to the root λ_3^+ will, therefore, be taken into consideration only for discontinuity surfaces of the second kind ($U > 0$).

The nontrivial solutions of system (2.9) considered in the following can be written as

$$V_z^+ = A, \quad V^+ = -iA, \quad P_1^+ = -d_1 (U + i\omega k^{-1}) A \qquad (\lambda = \lambda_1^+)$$

$$V_z^+ = B^{**}, \quad V^+ = -iB^{**}, \quad P_1^+ = 0 \qquad (\lambda = \lambda_3^+) \tag{2.11}$$

Here A and B^{**} are arbitrary constants, and the asterisks in the superscript denote that the constant does not vanish only in the case of a discontinuity surface of the second kind.

For the amplitude of perturbations in the region of a suspended layer we obtain from (2.5) the following system of equations:

$$\left(\frac{U}{\varepsilon}\lambda - i\omega\right) V_z^- = -\frac{\lambda}{d_1} P_1^- - \rho\sigma g_z \varepsilon \frac{W_z^- - V_z^-}{U}$$

$$\left(\frac{U}{\varepsilon}\lambda - i\omega\right) V^- = -\frac{ik}{d_1} P_1^- - \rho\sigma g_z \varepsilon \frac{W^- - V^-}{U}$$

$$-i\omega\rho W_z^- = -\frac{\lambda}{d_2}(\rho P_1^- + P_2^-) + \frac{\rho\sigma g_z \varepsilon^2}{1+\sigma} \frac{W_z^- - V_z^-}{U}$$

$$-i\omega\rho W^- = -\frac{ik}{d_2}(\rho P_1^- + P_2^-) + \frac{\rho\sigma g_z \varepsilon^2}{1+\sigma} \frac{W^- - V^-}{U}$$

$$\lambda V_z^- + ikV^- = 0, \quad \lambda W_z^- + ikW^- = 0 \tag{2.12}$$

The roots of the characteristic equation of system (2.12) are

$$\lambda_{1,2}^- = -k, \qquad \lambda_{3,4}^- = +k \tag{2.13}$$

$$\lambda_5^- = i\omega \frac{\varepsilon}{U} + \rho\sigma g_z \left(\frac{\varepsilon}{U}\right)^2 \left[1 - \left(1 + i\omega \frac{1+\sigma}{\sigma g_z} \frac{U}{\varepsilon^2}\right)^{-1}\right]$$

The solutions corresponding to roots $\lambda_{1,2}^-$ must be discarded, since they do not attenuate at infinity ($z \to -\infty$). Let us now consider the root λ_5^- and determine the sign of its real part. It can be shown that

$$\mathrm{Re}\,\lambda_5^- = -\frac{\varepsilon}{U}\mathrm{Im}\,\omega - \rho\frac{1+\sigma}{U}\frac{\mathrm{Im}\,\omega + \chi|\omega|^2}{1 + 2\chi\,\mathrm{Im}\,\omega + \chi^2|\omega|^2}, \qquad \chi = -\frac{1+\sigma}{\sigma g_z}\frac{U}{\varepsilon^2} \tag{2.14}$$

From (2.14) follows that perturbations which are either intensified with time ($\mathrm{Im}\,\omega > 0$) or are neutral ($\mathrm{Im}\,\omega = 0$) attenuate at infinity ($\mathrm{Re}\,\lambda_5^- > 0$) for $U < 0$ only, i.e. in the case of a discontinuity surface of the first kind. Hence, solutions corresponding to root λ_5^- should be taken into consideration only in this case; for a discontinuity surface of the second kind they must be discarded as either attenuating at infinity or a fortiori stable.

The nontrivial solutions of system (2.12) considered below may be written in the form

$$V_z^- = C, \qquad V^- = iC, \qquad W_z^- = D, \qquad W^- = iD$$

$$P_1^- = \frac{d_1}{k}\left(i\omega - \frac{U}{\varepsilon}k + \rho\sigma g_z\frac{\varepsilon}{U}\right)C - \rho\sigma g_z\frac{\varepsilon}{U}\frac{d_1}{k}D$$

$$P_2^- = \frac{d_1\rho}{k}\left(-i\omega + \frac{U}{\varepsilon}k - \sigma g_z\frac{\varepsilon}{U}\right)C + \frac{d_2\rho}{k}\left(i\omega + \frac{\sigma g_z}{1+\sigma}\frac{\varepsilon}{U}\right)D$$

$$(\lambda = \lambda_{3,4}^-)$$

$$V_z^- = E^*, \qquad V^- = i\lambda_5^- k^{-1}E^*, \qquad P_1^- = P_2^- = 0$$

$$W_z^- = \left[1 + \frac{1}{\rho\sigma g_z}\frac{U}{\varepsilon}\left(i\omega - \lambda_5^-\frac{U}{\varepsilon}\right)\right]E^*$$

$$W^- = i\left[1 + \frac{1}{\rho\sigma g_z}\frac{U}{\varepsilon}\left(i\omega - \lambda_5^-\frac{U}{\varepsilon}\right)\right]\frac{\lambda_5^-}{k}E^* \tag{2.15}$$

$$(\lambda = \lambda_5^-)$$

Here C, D and E^* are arbitrary constants, while the asterisk in the superscript denotes the constant which is not zero only in the case of a discontinuity surface of the first kind.

Substituting the spectral expansion (2.7) into the condition (2.6) at the discontinuity surface, and using transformations (2.8) and solutions (2.11) and (2.15), we obtain for the constants A, B^{**}, C, D and E^* and for the amplitude of perturbations of the discontinuity surface Z the following system of equations:

$$D + \left[1 + \frac{1}{\rho\sigma g_z}\frac{U}{\varepsilon}\left(i\omega - \lambda_5^-\frac{U}{\varepsilon}\right)\right]E^* + i\omega Z = 0 \tag{2.16}$$

$$A + B^{**} - \varepsilon C - \varepsilon E^* + i\omega\rho Z = 0, \quad A + B^{**} + C + \frac{\lambda_5^-}{k}E^* + \rho\frac{U}{\varepsilon}kZ = 0$$

$$\left(i\omega + \frac{U}{\varepsilon^2}k\right)A + U\left(\frac{1}{\varepsilon^2} - 1\right)kB^{**} + \left(i\omega - \frac{U}{\varepsilon}k + \rho\sigma g_z\frac{\varepsilon}{U}\right)C -$$
$$- \rho\sigma g_z\frac{\varepsilon}{U}D + \left[i\omega U\left(\frac{1}{\varepsilon^2} - 1\right) + \rho\sigma g_z\right]kZ = 0$$

$$\rho\frac{U}{\varepsilon^2}kA + \rho\frac{U}{\varepsilon^2}kB^{**} + \left(i\omega - \frac{U}{\varepsilon}k + \sigma g_z\frac{\varepsilon}{U}\right)C -$$
$$- \left[i\omega(1+\sigma) + \sigma g_z\frac{\varepsilon}{U}\right]D + \left(i\omega\rho\frac{U}{\varepsilon^2}k + \frac{\alpha}{d_1\rho}k^3\right)Z = 0$$

We thus have a homogeneous system of five equations in five unknowns (we remind that in each case only one of the values with an asterisk in the superscript differs from zero). The condition of existence of nontrivial solutions of system (2.16) yields the equation for ω. The looked for condition of stability is derived from the analysis of the sign of the imaginary part of the roots of this equation.

Let us consider separately the two variants of the discontinuity surface.

D i s c o n t i n u i t y   s u r f a c e   o f   t h e   f i r s t   k i n d. When the suspended layer lies over a homogeneous dispersing medium, $g_z = +g$, $U < 0$ and $B^{**} = 0$. The characteristic equation of system (2.16) is a fifth power algebraic equation in ω. The analysis of its roots is considerably simplified if one takes into consideration that the left-hand side of this equation can be written as the product of second and third order polynomials. The characteristic equation can be written as

$$\Pi_2(\Omega)\,\Pi_3(\Omega) = 0, \qquad \Omega = -i\omega \tag{2.17}$$

$$\Pi_2(\Omega) = \Omega^2 + \left(\frac{U}{\varepsilon}k - \frac{\varepsilon\sigma g}{U}\,\frac{1+\rho\sigma}{1+\sigma}\right)\Omega - \frac{\varepsilon\sigma g}{1+\sigma}\,k$$

$$\Pi_3(\Omega) = a_0\Omega^3 + a_1\Omega^2 + a_2\Omega + a_3$$

$$a_0 = 2 + (1+\varepsilon)\,\sigma, \qquad a_1 = -\frac{U}{\varepsilon}\,a_0 k - \frac{\varepsilon}{U}(2+\rho\sigma)\,\sigma g$$

$$a_2 = \frac{\alpha(1+\varepsilon)}{\rho d_1}k^3 + \frac{\rho}{\varepsilon}U^2k^2 + (3\varepsilon - 1)\,\sigma g k$$

$$a_3 = -\frac{\alpha\varepsilon\sigma g}{d_1 U}k^3\left(\frac{1+\varepsilon}{\rho\varepsilon^2\sigma g}U^2k + 1\right) - \frac{\rho}{\varepsilon^2}U^3k^3 + \frac{1+\varepsilon}{\varepsilon}\rho\sigma g U k^2 + \rho\varepsilon\sigma^2 g^2\frac{k}{U}$$

It is now clear from (2.13) that the roots of polynomial $\Pi_2(\Omega)$ reduce the difference $\lambda_5^- - k$ to zero. But, then, it follows from (2.16) and (2.15) that, when none of these roots is a root of polynomial $\Pi_3(\Omega)$, the corresponding solutions of system (2.16) are such that all perturbations are of zero amplitude. Hence it is sufficient to consider only the roots of polynomial $\Pi_3(\Omega)$.

The definition of Ω implies a negative $\operatorname{Re}\Omega$ as the condition of stability. Applying the Hurwitz criterion to polynomial $\Pi_3(\Omega)$, we obtain the following necessary and sufficient conditions for stability:

$$a_0 > 0, \qquad a_1a_2 - a_0a_3 > 0, \qquad a_3 > 0 \tag{2.18}$$

It will be seen from (2.17) that the first of these conditions is independent of the wave number k and is always satisfied. The second condition may be written as

$$\frac{\alpha}{\rho d_1}k^2 + \frac{1+\rho+\sigma}{\varepsilon^2}U^2k + (1+\rho\sigma)\,\sigma g > 0$$

Clearly, this condition is satisfied for any k (we recall that by definition $k \geqslant 0$). The remaining third condition is conveniently presented in the dimensionless form

$$\Lambda^3 + \frac{1+\varepsilon}{\varepsilon^2}\Lambda^2 - \left(\Sigma^2 + \frac{1}{\varepsilon^3}\right)\Lambda - \frac{1}{\varepsilon^2}\,\frac{1+\varepsilon}{1-\varepsilon}\,\Sigma^2 < 0 \tag{2.19}$$

$$\Lambda = \frac{\sigma g}{U^2 k}, \qquad \Sigma = \left(\frac{\alpha\sigma g}{\rho d_1 U^4}\right)^{1/2}$$

Hence the discontinuity surface is stable for all wave numbers satisfying the inequality

$$k > k_* = \frac{\sigma g}{U^2\Lambda_*} \tag{2.20}$$

where Λ_* is the only one positive root of the polynomial at the left-hand side of inequality (2.19). When $k < k_*$, the discontinuity is unstable.

Discontinuity surface of the second kind. In this case the homogeneous dispersing medium lies above the suspended layer, and $g_z = -g$, $U > 0$ and $E^* = 0$. The characteristic equation of system (2.16) may be written as

$$(\Omega - Uk)\left[\left(1 + \frac{1+\varepsilon}{2}\sigma\right)\Omega^2 + \sigma g\frac{\varepsilon}{U}\Omega + \frac{\alpha(1+\varepsilon)}{2\rho d_1}k^3 + \frac{\rho}{2\varepsilon}U^2k^2 + \frac{\rho}{2}\sigma g k\right] = 0, \qquad \Omega = -i\omega \tag{2.21}$$

The root $\Omega = Uk$ corresponds to $\lambda_3^+ = -k$ and, as in the case of a discontinuity surface of the first kind, the amplitudes of all perturbations are zero. The other two roots of the characteristic equation have, obviously, negative real parts for any $k > 0$.

Thus the discontinuity surface considered here is always stable with perturbations of any frequency, and this agrees with experimental data. Since this result, as evinced by (2.21), is independent of the value of surface tension, it is not surprising that Murray [6] had arrived at the correct conclusion in spite of incorrectly stipulated boundary conditions at the discontinuity surface.

In conclusion we would note that the results obtained here can be extended by a similar method to the case of nonlinear resistance of particles and of a suspended layer of finite thickness.

BIBLIOGRAPHY

1. Davidson, I. F. and Harrison, D., Fluidization of Solid Particles (Russian translation), M., "Khimiia", 1965.
2. Gel'perin, N. I., Ainshtein, V. G. and Kvasha, V. B., Fundamentals of the technique of fuidization. M., "Khimiia", 1967.
3. Rakhmatulin, Kh. A., Fundamentals of the gasdynamics of interpenetrating motions of compressible media. PMM Vol. 20, №2, 1956.
4. Kleiman, Ia. Z., On the propagation of strong discontinuities in a multicomponent medium. PMM Vol. 22, №2, 1958.
5. Rice, W. J. and Wilhelm, R. H., Surface dynamics of fluidized beds and quality of fluidization. A.I.Ch.E.J., Vol. 4, №4, 1958.
6. Murray, J. D., On the mathematics of fluidization. Pt. 1: Fundamental equations and wave propagation. J. Fluid Mech., Vol. 21, №3, 1965.
7. Furukawa, J. and Ohmae, T., Liquid-like properties of fluidized systems. Industr. and Engrg. Chem., Vol. 50, №5, 1958.
8. Landau, L. D. and Lifshits, E. M., Mechanics of Continuous Media. M., Gostekhizdat, 1954.
9. Goroshko, V. D., Rozenbaum, R. B. and Todes, O. M., Approximate laws of hydraulics of suspended layers and constrained fall. Izv. Vuz., Neft' i Gaz., №1, 1958.
10. Richardson, J. F. and Zaki, W. N., Sedimentation and fluidization, Pt. 1. Trans. Inst. Chem. Engrs., Vol. 32, №1, 1954.

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